THE HARISH-CHANDRA ISOMORPHISM FOR CLIFFORD ALGEBRAS

YURI BAZLOV

Abstract. We study an analogue of the Harish-Chandra homomorphism where the universal enveloping algebra $U(\mathfrak{g})$ is replaced by the Clifford algebra, $C\ell(\mathfrak{g})$, of a semisimple Lie algebra $\mathfrak{g}$. Two main goals are achieved. First, we prove that there is a Harish-Chandra type isomorphism between the subalgebra of $\mathfrak{g}$-invariants in $C\ell(\mathfrak{g})$ and the Clifford algebra $C\ell(\mathfrak{h})$ of the Cartan subalgebra of $\mathfrak{g}$. Second, the Cartan subalgebra $\mathfrak{h}$ is identified, via this isomorphism, with the graded space of the so-called primitive skew-symmetric invariants of $\mathfrak{g}$. The grading leads to a distinguished orthogonal basis of $\mathfrak{h}$, which turns out to be induced from the Langlands dual Lie algebra $\mathfrak{g}^\vee$ via the action of its principal three-dimensional subalgebra. This settles a conjecture of Kostant.

Introduction

Introduced by Harish-Chandra more than half a century ago, the Harish-Chandra homomorphism is of utmost significance in representation theory of semisimple Lie groups and algebras; character theory is one of the areas where it plays a key role. Recall that, given a complex semisimple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$, the Harish-Chandra homomorphism is a one-to-one algebra map between the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and the algebra $S(\mathfrak{h})^W$ of (translated) Weyl group W invariant polynomial functions on the space $\mathfrak{h}^*$. Because W is a finite reflection group, by the well-known Chevalley-Shephard-Todd theorem $S(\mathfrak{h})^W$ is a polynomial algebra. The Harish-Chandra map thus establishes the polynomiality of the algebra $Z(\mathfrak{g})$ and identifies the characters of $Z(\mathfrak{g})$ with W-orbits in $\mathfrak{h}^*$.

The subject of the present paper is a natural analogue of the Harish-Chandra homomorphism where $U(\mathfrak{g})$ is replaced with the Clifford algebra, $C\ell(\mathfrak{g})$, of a semisimple Lie algebra $\mathfrak{g}$. This analogue is based on a remarkably easy algebraic construction, which underlines the "classical" Harish-Chandra homomorphism but applies to a class of algebras much wider than that of universal enveloping algebras $U(\mathfrak{g})$. Let A be an associative algebra that factorises as $A_-\otimes A_0\otimes A_+$, where $A_\pm$ and A_0 are subalgebras in A and the tensor product is realised by the multiplication in A; we stress that A_-, A_0 and A_+ need not commute in A. Some further conditions on this factorisation guarantee that there exists a projection (in general, not an algebra map) $pr = \varepsilon_-\otimes \mathrm{id}\otimes\varepsilon_+$ of A onto its subalgebra A_0. (General details are discussed in [5].)

In the case $A = U(\mathfrak{g})$, the standard triangular decomposition $\mathfrak{g} = \mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ of a semisimple Lie algebra $\mathfrak{g}$ gives rise to a factorisation as above with $A_\pm = U(\mathfrak{n}_\pm)$ and $A_0 = U(\mathfrak{h}) = S(\mathfrak{h})$ the polynomial algebra. The Harish-Chandra map is obtained by restricting pr to $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$, the $ad\,\mathfrak{g}$-invariants in $U(\mathfrak{g})$. (This method also works for the quantised universal enveloping algebra $U_\hbar(\mathfrak{g})$ [25].) The same approach is used in [26] to give a completely algebraic construction of the homomorphism $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}\to\mathcal{D}(\mathfrak{h})^W$, also due to Harish-Chandra (here $\mathcal{D}(\cdot)$ stands for polynomial differential operators).

The Clifford algebra $C\ell(\mathfrak{g})$ factorises as a product of three of its subalgebras, $C\ell(\mathfrak{n}_-)$, $C\ell(\mathfrak{h})$ and $C\ell(\mathfrak{n}_+)$, therefore admitting a Harish-Chandra map $\Phi\colon C\ell(\mathfrak{g})\to C\ell(\mathfrak{h})$. Studying the map Φ involves

structural theory of the Clifford algebra $C\ell(\mathfrak{g})$, based mostly on results of Kostant. In the present paper, we focus on the restriction of the Harish-Chandra map Φ to the subalgebra $J = C\ell(\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants, which in the Clifford algebra plays a role similar to that of the centre $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ in $U(\mathfrak{g})$. Indeed, Kostant's "separation of variables" result [19] states that $C\ell(\mathfrak{g}) = E \otimes J$ is a free module over the algebra J, and J is further described as a Clifford algebra $C\ell(P)$ of a remarkable space P of the so-called primitive invariants.

Our first main result, Theorem 4.1, shows that Φ restricts to an isomorphism $J \xrightarrow{\sim} C\ell(\mathfrak{h})$. Comparing this to the universal enveloping algebra, it is worth noting that the action of the Weyl group W on $C\ell(\mathfrak{h})$ is conspicuously absent from the picture. We then refine the isomorphism result by showing that the Harish-Chandra map Φ identifies the space P of primitive invariants with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The above immediately raises the next question: the space P is naturally graded, via its inclusion in the exterior algebra $\bigwedge\mathfrak{g}$ and its identification with primitive homology classes in the homology of $\mathfrak{g}$; what is the grading induced on the Cartan subalgebra by the Harish-Chandra map? It turns out that the answer involves the so-called Langlands dual Lie algebra $\mathfrak{g}^\vee$ of $\mathfrak{g}$. This is a complex semisimple Lie algebra with a root system dual to that of $\mathfrak{g}$. There is a canonical copy of $\mathfrak{sl}_2$ inside $\mathfrak{g}^\vee$, and its adjoint action on $\mathfrak{g}^\vee$ splits $\mathfrak{g}^\vee$ into a direct sum of $\mathfrak{sl}_2$-submodules ("strings"). Intersections of these "strings" with the Cartan subalgebra $\mathfrak{h}$, which is viewed as shared between $\mathfrak{g}$ and $\mathfrak{g}^\vee$, define the graded components of $\mathfrak{h}$. This is established in Theorem 5.5 which is the second and final main result of the paper. In particular, Theorem 5.5 confirms a conjecture made by Kostant [20].

When $\mathfrak{g}$ is a simple Lie algebra, the graded components in $\mathfrak{h}$ typically are one-dimensional, giving rise to a distinguished basis of $\mathfrak{h}$. (The only deviation from this occurs in a simple Lie algebra with Dynkin diagram of type D on an even number of nodes.) This basis is orthogonal with respect to the Killing form and contains ρ, half the sum of positive roots of $\mathfrak{g}$; we refer to this basis of $\mathfrak{h}$ as the (Langlands dual) principal basis of the Cartan subalgebra.

Acknowledgments. Section 4 of the present paper is based on a revision of work done in my PhD thesis [3], which was completed with the guidance of Anthony Joseph. I am grateful to Bertram Kostant for conversations on the subject of this paper; having seen partial results of [3], he suggested a conjectural link to principal TDS bases, proved in the present paper. I thank Anton Alekseev for providing helpful ideas for the proof of Theorem 5.5. These useful ideas and calculations are also made clear in a recent preprint [23] by R. P. Rohr. I am grateful to Victor Ginzburg and Eckhard Meinrenken for their comments on earlier versions of this work and to David Kazhdan for a stimulating discussion.

Contents

1. The classical Chevalley projection and Harish-Chandra map

We start by recalling the Chevalley projection map and the Harish-Chandra map associated with a semisimple Lie algebra $\mathfrak{g}$. This will serve as an introduction to the subsequent treatment of the analogues of these for skew-symmetric tensors.

1.1. **Invariant symmetric tensors and the Chevalley projection.** Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r over the complex field $\mathbb{C}$. Let a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, such that $\mathfrak{h} \subset \mathfrak{b}$, be fixed. This partitions the root system of $\mathfrak{g}$ into positive and negative parts, and gives rise to the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

of $\mathfrak{g}$, where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are subspaces of $\mathfrak{g}$ spanned by root vectors corresponding to negative and positive roots, respectively. By $S(\mathfrak{g})$ we denote the algebra of symmetric tensors over $\mathfrak{g}$, which is the same as the algebra of polynomial functions on the space $\mathfrak{g}^*$. It is graded by degree: $S(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g})$, and we identify $\mathfrak{g}$ with $S^1(\mathfrak{g})$. The action of $\mathfrak{g}$ on $S(\mathfrak{g})$ is extended from the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} = S^1(\mathfrak{g})$ by derivations of degree 0; we denote the action of $x \in \mathfrak{g}$ by $ad\, x \in \operatorname{End} S(\mathfrak{g})$. We refer to the set

$$J_S = S(\mathfrak{g})^{\mathfrak{g}} = \{f \in S(\mathfrak{g}) \mid (ad\, x)f = 0 \ \forall x \in \mathfrak{g}\}$$

as the space of symmetric $\mathfrak{g}$-invariants. Note that J_S is a graded subalgebra of $S(\mathfrak{g})$. It is obvious that the algebra $S(\mathfrak{g})$ has triangular factorisation,

$$S(\mathfrak{g}) \cong S(\mathfrak{n}_-) \otimes S(\mathfrak{h}) \otimes S(\mathfrak{n}_+),$$

into three subalgebras generated by $\mathfrak{n}_-$, $\mathfrak{h}$, $\mathfrak{n}_+$, respectively. Denote by ε the character of an algebra of polynomials given by the evaluation of a polynomial at zero. The homomorphism

$$\Psi_0 = \varepsilon \otimes \mathrm{id} \otimes \varepsilon \colon S(\mathfrak{g}) \to S(\mathfrak{h})$$

of commutative algebras is what is typically called the Chevalley projection map. By a classical result of Chevalley, see [10, Theorem 7.3.7], the restriction of Ψ_0 to J_S is a graded algebra isomorphism

$$\Psi_0 \colon J_S \xrightarrow{\sim} S(\mathfrak{h})^W.$$

Here $S(\mathfrak{h})^W$ denotes symmetric tensors over $\mathfrak{h}$ invariant under the action of the Weyl group W of $\mathfrak{g}$ (this action is extended from $\mathfrak{h}$ to $S(\mathfrak{h})$).

1.2. **The classical Harish-Chandra map.** Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and

$$J_U = U(\mathfrak{g})^{\mathfrak{g}}$$

be its centre, which is the ring of invariants of the adjoint representation of $\mathfrak{g}$ in $U(\mathfrak{g})$. The Poincaré-Birkhoff-Witt symmetrisation map

$$\beta \colon S(\mathfrak{g}) \to U(\mathfrak{g})$$

is a $\mathfrak{g}$-module isomorphism between $S(\mathfrak{g})$ and $U(\mathfrak{g})$, where $\beta(x_1 x_2 \dots x_n)$ for $x_i \in \mathfrak{g}$ is defined as $\frac{1}{n!}$ times the sum of $x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(n)} \in U(\mathfrak{g})$ over all permutations π in n letters. In particular, β identifies J_S and J_U as linear spaces. Note that β is not an algebra isomorphism between J_S and J_U. (An algebra isomorphism $J_S \xrightarrow{\sim} J_U$, known as the Duflo map, was explicitly constructed in [11].) The Poincaré-Birkhoff-Witt theorem for $U(\mathfrak{g})$ implies the triangular factorisation

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$$

of the algebra $U(\mathfrak{g})$ into the subalgebras generated by $\mathfrak{n}_-$, $\mathfrak{h}$, $\mathfrak{n}_+$, respectively. Let $\varepsilon_-\colon U(\mathfrak{n}_-)\to\mathbb{C}$ be the algebra homomorphism defined by $\varepsilon_-(x)=0$ for $x\in\mathfrak{n}_-$, and let $\varepsilon_+\colon U(\mathfrak{n}_+)\to\mathbb{C}$ be defined similarly. We will refer to the map

$$\Psi=\varepsilon_-\otimes\mathrm{id}\otimes\varepsilon_+\colon U(\mathfrak{g})\to U(\mathfrak{h})$$

as the Harish-Chandra map, slightly abusing the terminology. As $\mathfrak{h}$ is an Abelian Lie algebra, $U(\mathfrak{h})$ is identified with the polynomial algebra $S(\mathfrak{h})$. The map Ψ is not an algebra homomorphism, but its restriction to the subalgebra $U(\mathfrak{g})^{\mathfrak{h}}=\{f\in U(\mathfrak{g})\mid (ad\,h)f=0\ \forall h\in\mathfrak{h}\}$ is. Further restricting Ψ to J_U which is a subalgebra of $U(\mathfrak{g})^{\mathfrak{h}}$, one obtains the isomorphism

$$\Psi\colon J_U\xrightarrow{\sim}S(\mathfrak{h})^{W\cdot}$$

between the centre of $U(\mathfrak{g})$ and the ring $S(\mathfrak{h})^{W\cdot}$ of symmetric tensors over $\mathfrak{h}$ invariant under the shifted action of W. (The shifted action of $w\in W$ is defined on $\lambda\in\mathfrak{h}^*$ by $w.\lambda=w(\lambda+\rho)-\rho$ where $\rho\in\mathfrak{h}^*$ is the half sum of positive roots of $\mathfrak{g}$, and hence on $S(\mathfrak{h})$ which is viewed as the algebra of polynomial functions on $\mathfrak{h}^*$.) This isomorphism is due to Harish-Chandra; see [15], [10, Theorem 7.4.5].

1.3. **An analogue of the Chevalley projection for $\bigwedge\mathfrak{g}$.** We are going to consider $\bigwedge\mathfrak{g}$, the exterior algebra of $\mathfrak{g}$, as a "skew-symmetric analogue" of $S(\mathfrak{g})$. Furthermore, the universal enveloping algebra $U(\mathfrak{g})$ — a deformation of $S(\mathfrak{g})$ — will be replaced by $C\ell(\mathfrak{g})$, the Clifford algebra of $\mathfrak{g}$, which is a deformation of $\bigwedge\mathfrak{g}$. We will now discuss the analogue of the Chevalley projection Ψ_0 (and later, of the Harish-Chandra map Ψ) in the skew-symmetric situation.

The action of $\mathfrak{g}$ on the finite-dimensional algebra $\bigwedge\mathfrak{g}=\bigoplus_{n=0}^{\dim\mathfrak{g}}\bigwedge^n\mathfrak{g}$ is an extension of the adjoint action of $\mathfrak{g}$ as a derivation of degree 0. Let us denote the action of $x\in\mathfrak{g}$ by $\theta(x)\in\mathrm{End}(\bigwedge\mathfrak{g})$; that is, $\theta(x)y=[x,y]$ where $y\in\wedge^1\mathfrak{g}=\mathfrak{g}$, and $\theta(x)(u\wedge v)=(\theta(x)u)\wedge v+u\wedge\theta(x)v$ for $u,v\in\bigwedge\mathfrak{g}$. The subspace

$$J=(\textstyle\bigwedge\mathfrak{g})^{\mathfrak{g}}=\{u\in\bigwedge\mathfrak{g}\mid\theta(x)u=0\ \forall x\in\mathfrak{g}\},$$

of $\theta(\mathfrak{g})$ invariants (the invariant skew-symmetric tensors over $\mathfrak{g}$) is a graded $\wedge$-subalgebra of $\bigwedge\mathfrak{g}$. The triangular factorisation

$$\textstyle\bigwedge\mathfrak{g}=\bigwedge\mathfrak{n}_-\otimes\bigwedge\mathfrak{h}\otimes\bigwedge\mathfrak{n}_+$$

and the "augmentation maps" $\varepsilon_\pm\colon\bigwedge\mathfrak{n}_\pm\to\mathbb{C}$ which are algebra homomorphisms uniquely defined by $\varepsilon_\pm(\mathfrak{n}_\pm)=0$, give rise to a degree-preserving projection map

$$\Phi_0=\varepsilon_-\otimes\mathrm{id}\otimes\varepsilon_+\colon\textstyle\bigwedge\mathfrak{g}\to\bigwedge\mathfrak{h}\ .$$

Let us consider the restriction of Φ_0 to J. In contrast to the Chevalley projection map for symmetric tensors, this restriction fails to be one-to-one: one has

$$\Phi_0(J)=\textstyle\bigwedge^0\mathfrak{h}=\mathbb{C}.$$

Indeed, one shows that the Φ_0-image of J must lie in the W-invariants in $\bigwedge\mathfrak{h}$. However, the fixed points of W in $\bigwedge\mathfrak{h}$ are just the one-dimensional space $\bigwedge^0\mathfrak{h}$; see [13, Section 5.1].

Although the injectivity of the Chevalley projection Φ_0 on $\bigwedge\mathfrak{g}$ fails so miserably, passing to its counterpart Φ (a Clifford algebra version of the Harish-Chandra map) rectifies the situation, as we will discover in due course.

2. Clifford algebras

In this Section, we recall some basics on Clifford algebras associated to quadratic forms on complex vector spaces, define the algebra $C\ell(\mathfrak{g})$, and introduce the Clifford algebra version of the Harish-Chandra map.

2.1. **Identification of $C\ell(V)$ with $\bigwedge V$.** Let V be a finite-dimensional vector space over $\mathbb{C}$, equipped with a symmetric bilinear form $(\ ,\ )$. The Clifford algebra $C\ell(V)$ of V is the quotient of the full tensor algebra $T(V)$ modulo the two-sided ideal generated by $\{x \otimes x - (x,x) \mid x \in V\}$. We will denote the Clifford product of $u, v \in C\ell(V)$ by $u \cdot v$. An isomorphic image of the space $V = T^1(V)$ is contained in $C\ell(V)$; one has $xy + yx = 2(x,y)$ for $x, y \in V$.

A convenient point of view that we adhere to in the present paper is that the Clifford algebra $C\ell(V)$ has the same underlying linear space as $\bigwedge V$, but the Clifford product is a deformation of the exterior product. Explicitly, following [19], for $x \in V$ let the operator $\iota(x)\colon \bigwedge^1 V \to \bigwedge^0 V$ be defined by $\iota(x)y = (x,y)$ where $y \in V$. Extend $\iota(x)$ to a superderivation of $\bigwedge V$ of degree -1, i.e., by the rule

$$\iota(x)(u \wedge v) = (\iota(x)u) \wedge v + (-1)^{|u|} u \wedge \iota(x)v, \qquad u \in \textstyle\bigwedge^{|u|} V,\ v \in \bigwedge V.$$

We refer to $\iota(x)\colon \bigwedge V \to \bigwedge V$ as the contraction operator associated to $x \in V$. Now define the operator $\gamma(x) \in \mathrm{End}(\bigwedge V)$ by $\gamma(x)u = x \wedge u + \iota(x)u$, $u \in \bigwedge \mathfrak{g}$. The superderivation property of $\iota(x)$ and the fact that $\iota(x)^2 = 0$ and $x \wedge x = 0$ imply that $\gamma(x)^2$ is multiplication by the scalar (x,x). Therefore, γ extends to a homomorphism $\gamma\colon C\ell(V) \to \mathrm{End}(\bigwedge V)$. The linear map

$$\sigma\colon C\ell(V) \to \textstyle\bigwedge V, \qquad \sigma(u) = \gamma(u)1,$$

is bijective, cf. [8, Theorem II.1.6]. It is this map σ that is used to identify the underlying linear space of $C\ell(V)$ with $\bigwedge V$.

Remark 2.1. The symmetric algebra $S(V)$ and the exterior algebra $\bigwedge V$ of a vector space V are the simplest examples of the so-called Nichols algebras (terminology introduced by Andruskiewitsch and Schneider). Nichols algebras are Hopf algebras in a braided category; see [22, 27, 2, 4] for background. In particular, Nichols algebra $\mathcal{B}(V)$ of a braided space V (a space equipped with an invertible operator $c \in \mathrm{End}(V \otimes V)$ satisfying the quantum Yang-Baxter equation) has a set of braided derivations ∂_ξ, indexed by $\xi \in V^*$, that satisfy the braided Leibniz rule inferred from the braiding c. One takes the braiding $c(x \otimes y) = y \otimes x$, respectively $c(x \otimes y) = -y \otimes x$, to obtain the Nichols algebra $S(V)$, respectively $\bigwedge V$, of V. Via the map $V \to V^*$ given by the form $(\ ,\ )$, the partial derivative $\frac{\partial}{\partial x}$ and the contraction operator $\iota(x)$, $x \in V$, are braided derivations of $S(V)$ and $\bigwedge V$, respectively.

We also remark that the inverse to the map $\sigma\colon C\ell(V) \to \bigwedge V$ can be written, in a completely different fashion, as the skew-symmetrisation map

$$\sigma^{-1} = \beta_\wedge\colon \textstyle\bigwedge V \to C\ell(V), \qquad \beta_\wedge(x_1 \wedge \dots \wedge x_n) = \dfrac{1}{n!}\displaystyle\sum_\pi (\mathrm{sgn}\ \pi) x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(n)},$$

where $x_i \in V$ and the sum on the right is over all permutations π of the indices $1, \dots, n$.

2.2. **Algebras $C\ell_\hbar(V)$.** To emphasise our earlier point that the Clifford product on $\bigwedge V$ is a deformation of the wedge product, and for later use in calculations, we introduce a complex-valued deformation parameter $\hbar$. To each value of $\hbar$ we associate a bilinear form $(x,y)_\hbar := \hbar \cdot (x,y)$ on V. Denote by $C\ell_\hbar(V)$ the Clifford algebra of the form $(\ ,\ )_\hbar$. Observe that, for $\hbar \neq 0$, the algebra

$C\ell_\hbar(V)$ is isomorphic to the original Clifford algebra $C\ell(V) = C\ell_\hbar(V)|_{\hbar=1}$, because the linear map $V \to V$, $x \mapsto \hbar^{1/2}x$, extends to an algebra isomorphism $C\ell_\hbar(V) \to C\ell(V)$. On the other hand, $C\ell_\hbar(V)|_{\hbar=0}$ coincides with the exterior algebra $\bigwedge V$ and is not isomorphic to $C\ell(V)$ unless the form $(\ ,\ )$ is identically zero.

This construction gives rise to a family of Clifford products $\{\cdot_\hbar \mid \hbar \in \mathbb{C}\}$ on the space $\bigwedge V$. The way the product $a \cdot_\hbar b$ depends on $\hbar$ is described in

Lemma 2.2. *If $a \in \bigwedge^i V$, $b \in \bigwedge^j V$ are homogeneous elements in the exterior algebra of V, there exist $u_{i+j-2s} \in \bigwedge^{i+j-2s} V$ for $s = 1, 2, \ldots, \lfloor \frac{i+j}{2} \rfloor$ such that*

$$a \cdot_\hbar b = a \wedge b + \hbar\ u_{i+j-2} + \hbar^2 u_{i+j-4} + \cdots = a \wedge b + \sum_{1 \le s \le (i+j)/2} \hbar^s u_{i+j-2s}.$$

Proof. The statement is proved by induction in i, the degree of a. If $i = 0$, put $u_{0+j-2s} = 0$ for all s. If $i = 1$ and $a = x \in V$, then $a \cdot_\hbar b = x \wedge b + \hbar\, \iota(x)b$. In this case, put $u_{1+j-2} = \iota(x)b$ and $u_{1+j-2s} = 0$ for all $s > 1$.

To prove the assertion for $i \ge 2$, one may assume that $a = x \wedge a'$ for some $x \in V$ and $a' \in \bigwedge^{i-1} V$. Put $a'' = \hbar\, \iota(x)a'$ so that the element a'' is homogeneous of degree $i-2$ in $\bigwedge V$. Then $a = x \cdot_\hbar a' - a''$, hence $a \cdot_\hbar b = x \cdot_\hbar (a' \cdot_\hbar b) - a'' \cdot_\hbar b$. By the induction hypothesis, the Clifford products $a' \cdot_\hbar b$ and $a'' \cdot_\hbar b$ have required expansions in $\bigwedge V$. It only remains to apply the $i = 1$ case to the product $x \cdot_\hbar (a' \cdot_\hbar b)$ and to collect the terms, which gives the required expansion for $a \cdot_\hbar b$. $\square$

2.3. **The superalgebra structure on $C\ell(V)$.** Clearly, the Clifford product on the exterior algebra $\bigwedge V$ does not respect the grading on $\bigwedge V$, and $C\ell(V)$ is not a graded algebra. It is, however, easy to see (and is apparent from Lemma 2.2) that $C\ell(V)$ is still a superalgebra:

$$C\ell(V) = C\ell^{\bar{0}}(V) \oplus C\ell^{\bar{1}}(V),$$

with $C\ell^i(V) = \sum_{n \ge 0} \bigwedge^{2n+i} V$ for $i = \bar{0}, \bar{1}$ (residues modulo 2). For later use, we observe the fact that the contraction operators $\iota(x)\colon \bigwedge V \to \bigwedge V$ are superderivations with respect to the Clifford multiplication:

Lemma 2.3. *For any $i \in \{\bar{0}, \bar{1}\}$, $x \in V$, $u \in C\ell^i(V)$ and $v \in C\ell(V)$,*

$$\iota(x)(u \cdot v) = (\iota(x)u) \cdot v + (-1)^i u \cdot \iota(x)v.$$

Proof. We have to show that $\iota(x)$ supercommutes with $\gamma(u)\colon \bigwedge V \to \bigwedge V$, the operator of the left Clifford multiplication by u. Since $\gamma(u)$ is in the subalgebra of $\mathrm{End}(\bigwedge V)$ generated by $\gamma(y)$, $y \in V$, it is enough to show that $\iota(x)$ supercommutes with $\gamma(y)$. Write $\gamma(y) = (y \wedge \cdot) + \iota(y)$. Now, $\iota(x)$ supercommutes with the operator $y \wedge \cdot$ of the left exterior multiplication by y, because $\iota(x)$ is a superderivation of the wedge product. Finally, to show that $\iota(x)$ supercommutes with $\iota(y)$, observe that, by a general fact about superderivations, the supercommutator $\iota(x)\iota(y) + \iota(y)\iota(x)$ must be an even superderivation of the wedge product; but it obviously vanishes on V, hence is identically zero on $\bigwedge V$. $\square$

2.4. **The Clifford algebra $C\ell(\mathfrak{g})$.** We are interested in the case when $V = \mathfrak{g}$ is a semisimple Lie algebra. We fix $(\ ,\ )$ to be a non-degenerate *ad*-invariant symmetric bilinear form on $\mathfrak{g}$. For example, $(\ ,\ )$ may be the Killing form, or be proportional to the Killing form with a non-zero coefficient. (If $\mathfrak{g}$ is simple, there are no other options.) We denote by $C\ell(\mathfrak{g})$ the Clifford algebra of $\mathfrak{g}$ with respect to the form $(\ ,\ )$.

Recall that by $\theta(g)$ is denoted the adjoint action of $g \in \mathfrak{g}$ on the exterior algebra $\bigwedge\mathfrak{g}$; one has $\theta(g)(x \wedge u) - x \wedge \theta(g)u = [g, x] \wedge u$ for all $g, x \in \mathfrak{g}$ and $u \in \bigwedge\mathfrak{g}$. Furthermore, it is easy to see that the ad-invariance of the form $(\ ,\ )$ implies $\theta(g)\iota(x) - \iota(x)\theta(g) = \iota([g, x])$. Thus, $\theta(g)$ is an (even) derivation of the Clifford product. It immediately follows that $\mathfrak{g}$-invariants $J = (\bigwedge\mathfrak{g})^{\mathfrak{g}}$ form a Clifford subalgebra in $C\ell(\mathfrak{g})$. Recall that J is also a wedge-subalgebra in $\bigwedge\mathfrak{g}$. We will elaborate on these two non-isomorphic algebra structures on J in the next Section.

2.5. **The Harish-Chandra map Φ for $C\ell(\mathfrak{g})$.** The central object of the paper is the following analogue of the Harish-Chandra map, defined for the Clifford algebra $C\ell(\mathfrak{g})$. Observe that $C\ell(\mathfrak{g})$, like all the algebras considered so far, factorises into its subalgebras, generated by the direct summands $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$ in the triangular decomposition of $\mathfrak{g}$. These subalgebras are themselves Clifford algebras that correspond to the restriction of the form $(\ ,\ )$ on the respective subspaces of $\mathfrak{g}$:

$$C\ell(\mathfrak{g}) = C\ell(\mathfrak{n}_-) \otimes C\ell(\mathfrak{h}) \otimes C\ell(\mathfrak{n}_+),$$

where the tensor product is realised by the Clifford multiplication (note that the tensorands do not commute with respect to Clifford multiplication). The subalgebras $C\ell(\mathfrak{n}_\pm)$ are supercommutative and are isomorphic to the exterior algebras $\bigwedge\mathfrak{n}_\pm$, because the restriction of $(\ ,\ )$ to $\mathfrak{n}_-$ (respectively to $\mathfrak{n}_+$) is necessarily zero. The restriction of $(\ ,\ )$ to the Cartan subalgebra $\mathfrak{h}$ is non-degenerate, thus $C\ell(\mathfrak{h})$ is a simple algebra or a direct sum of two simple algebras, much like the bigger algebra $C\ell(\mathfrak{g})$.

In line with all the previous definitions of Harish-Chandra type maps, introduce the Harish-Chandra map for $C\ell(\mathfrak{g})$ by

$$\Phi = \varepsilon_- \otimes \mathrm{id} \otimes \varepsilon_+ \colon C\ell(\mathfrak{g}) \to C\ell(\mathfrak{h}),$$

where $\varepsilon_\pm \colon C\ell(\mathfrak{n}_\pm) = \bigwedge\mathfrak{n}_\pm \to \mathbb{C}$ are the augmentation maps as above. Similar to the $U(\mathfrak{g})$ situation, the Harish-Chandra map Φ is not a homomorphism of algebras, but its restriction to $\mathfrak{h}$-invariants is:

Lemma 2.4. *The restriction of Φ to the subalgebra $C\ell(\mathfrak{g})^{\mathfrak{h}} = \{u \in C\ell(\mathfrak{g}) \mid \theta(\mathfrak{h})u = 0\}$ is a superalgebra homomorphism between $C\ell(\mathfrak{g})^{\mathfrak{h}}$ and $C\ell(\mathfrak{h})$.*

Proof. First of all, Φ is a map of superspaces: the triangular factorisation $C\ell(\mathfrak{g}) = C\ell(\mathfrak{n}_-) \otimes C\ell(\mathfrak{h}) \otimes C\ell(\mathfrak{n}_+)$ is compatible with the superspace structure on all tensorands, and moreover, the maps $\varepsilon_\pm \colon C\ell(\mathfrak{n}_\pm) \to \mathbb{C}$ are superspace maps (where $\mathbb{C}$ is a one-dimensional even space), and Φ is defined as $\varepsilon_- \otimes \mathrm{id} \otimes \varepsilon_+$ in this triangular factorisation.

Furthermore, write $L = \mathfrak{n}_- C\ell(\mathfrak{g})\mathfrak{n}_+ \subset C\ell(\mathfrak{g})$. Then $C\ell(\mathfrak{g})^{\mathfrak{h}} \subseteq 1 \otimes C\ell(\mathfrak{h}) \otimes 1 \oplus L$. The Lemma follows immediately from the fact that $L \cdot C\ell(\mathfrak{g})$, $C\ell(\mathfrak{g}) \cdot L$ are in the kernel of Φ. □

2.6. **The r-matrix formula for Φ.** Let us now express the Harish-Chandra map $\Phi \colon C\ell(\mathfrak{g}) \to C\ell(\mathfrak{h})$ in terms of the projection $\Phi_0 \colon \bigwedge\mathfrak{g} \to \bigwedge\mathfrak{h}$. For $i = 1, \ldots, n$, let x_i (respectively y_i) be the positive (respectively negative) root vectors in $\mathfrak{g}$, normalized so that $(x_i, y_i) = 1$. Denote

$$\mathfrak{r} = \sum_{i=1}^{n} x_i \wedge y_i \qquad \in \bigwedge\nolimits^2\mathfrak{g}.$$

The formula for $\mathfrak{r}$ is a standard way to write a classical skew-symmetric r-matrix of $\mathfrak{g}$. Now introduce the operator

$$\iota(\mathfrak{r}) \in \mathrm{End}(\textstyle\bigwedge\mathfrak{g}), \quad \iota(\mathfrak{r}) = \displaystyle\sum_{i=1}^{n} \iota(x_i)\iota(y_i),$$

of degree -2 with respect to the grading on $\bigwedge\mathfrak{g}$. This operator is used in the following

Proposition 2.5. *Modulo the identification of the spaces $C\ell(\mathfrak{g})$ and $\bigwedge\mathfrak{g}$, respectively $C\ell(\mathfrak{h})$ and $\bigwedge\mathfrak{h}$,*

$$\Phi(u) = \Phi_0(e^{\iota(\mathfrak{r})}u)$$

for any $u \in \bigwedge\mathfrak{g}$.

Proof. The algebra $\bigwedge\mathfrak{h}$ is viewed as an exterior and Clifford subalgebra of $\bigwedge\mathfrak{g}$. For any $u \in \bigwedge\mathfrak{h}$ one has $\Phi(u) = \Phi_0(u) = u$, and $\iota(\mathfrak{r})u = 0$ so that $e^{\iota(\mathfrak{r})}u = u$. Thus, both sides of the equation agree on $u \in C\ell(\mathfrak{h}) \subset C\ell(\mathfrak{g})$.

Observe, as in the proof of Lemma 2.4, that $C\ell(\mathfrak{g})^{\mathfrak{h}} \subset C\ell(\mathfrak{h}) + \mathfrak{n}_- \cdot C\ell(\mathfrak{g})$. Let us show that $\Phi(u) = \Phi_0(e^{\iota(\mathfrak{r})}u) = 0$ when $u \in \mathfrak{n}_- \cdot C\ell(\mathfrak{g})$. Of course, $\Phi(u) = 0$ simply by definition of the map Φ. Now, we may assume that $u \in y_j \cdot C\ell(\mathfrak{g})$ for some j between 1 and n; but since $\iota(\mathfrak{r})$ does not depend on a particular ordering of positive roots, we may assume j to be 1, i.e., $u = \gamma(y_1)u'$ for some $u' \in \bigwedge\mathfrak{g}$. Here $\gamma(y) = y \wedge \cdot + \iota(y)$ is the operator of left Clifford multiplication by y as in 2.1.

Note that the operators $\iota(x_i)\iota(y_i)$, $i = 1, \dots, n$, pairwise commute and square to zero. This follows from the fact that $\iota(x)$ and $\iota(y)$ anticommute for all $x, y \in \mathfrak{g}$, see the proof of Lemma 2.3. Hence, we may write $e^{\iota(\mathfrak{r})}$ as $\prod_{i=1}^n e^{\iota(x_i)\iota(y_i)} = \prod_{i=1}^n (1+\iota(x_i)\iota(y_i))$. Furthermore, because $(x_i, y_1) = (y_i, y_1) = 0$ for $i \neq 1$, it follows from Lemma 2.3 that $\iota(x_i)$ commutes with $\gamma(y_1)$ for $i \neq 1$. Thus,

$$\Phi_0(e^{\iota(\mathfrak{r})}u) = \Phi_0\big((1 + \iota(x_1)\iota(y_1))\gamma(y_1)u''\big)$$

for some $u'' \in \bigwedge\mathfrak{g}$. It remains to note that

$$(1 + \iota(x_1)\iota(y_1))z = \gamma(x_1)\gamma(y_1)z - x_1 \wedge y_1 \wedge z$$

and that, obviously, $\Phi_0(x_1 \wedge y_1 \wedge z) = 0$ for all $z \in \bigwedge\mathfrak{g}$. We are left with

$$\Phi_0(e^{\iota(\mathfrak{r})}u) = \Phi_0(\gamma(x_1)\gamma(y_1)^2 u'')$$

which is zero since $y_1^2 = (y_1, y_1) = 0$ in the Clifford algebra $C\ell(\mathfrak{g})$.

We have shown that both sides of the required equation agree when $u \in C\ell(g)^{\mathfrak{h}}$. Now suppose that u is an eigenvector of non-zero weight for the adjoint action of $\mathfrak{h}$ on $\bigwedge\mathfrak{g}$. Since the maps Φ, Φ_0 and $\iota(\mathfrak{r})$ preserve the weight with respect to the $\mathfrak{h}$-action, $\Phi(u)$ and $\Phi_0(e^{\iota(\mathfrak{r})}u)$ must have, in $\bigwedge\mathfrak{h}$, the same non-zero weight as u. Since the adjoint action of $\mathfrak{h}$ on $\bigwedge\mathfrak{h}$ is trivial, the latter is only possible if $\Phi(u) = \Phi_0(e^{\iota(\mathfrak{r})}u) = 0$. The Proposition is proved. □

Remark 2.6. The map $p_G^T \circ \mathcal{T}$ in [1, 3.1] coincides with $\Phi_0 \circ e^{\iota(\mathfrak{r})}$ (in our notation). Proposition 2.5 thus implies that the map $p_G^T \circ \mathcal{T}$ from [1] is the same as our Harish-Chandra map Φ.

Remark 2.7. Recall that for each non-zero value of the deformation parameter $\hbar$ we can equip $\bigwedge\mathfrak{g}$ with the structure of Clifford algebra $C\ell_\hbar(\mathfrak{g})$. The latter Clifford algebra is built with respect to the bilinear form $(\,,\,)_\hbar = \hbar \cdot (\,,\,)$ on $\mathfrak{g}$ which is *ad*-invariant and non-degenerate. In particular, the Harish-Chandra map

$$\Phi_\hbar \colon C\ell_\hbar(\mathfrak{g}) \to C\ell_\hbar(\mathfrak{h})$$

is defined. Let us apply Proposition 2.5 to the Clifford algebra $C\ell_\hbar(\mathfrak{g})$.

Namely, if x_i, y_i was a positive/negative root vector pair normalised by $(x_i, y_i) = 1$, then x_i, $\hbar^{-1}y_i$ will be such pair for the form $(\,,\,)_\hbar$. It follows that the classical r-matrix of $\mathfrak{g}$ corresponding to the new form $(\,,\,)_\hbar$ is given by $\mathfrak{r}_\hbar = \hbar^{-1}\mathfrak{r}$. Furthermore, the contraction operators on $\bigwedge\mathfrak{g}$ with respect to the bilinear form $(\,,\,)_\hbar$ are given by $\iota_\hbar(x) = \hbar\,\iota(x)$. Hence we have a new operator

$$\iota_\hbar(\mathfrak{r}_\hbar) := \sum_{i=1}^n \iota_\hbar(x_i)\iota_\hbar(\hbar^{-1}y_i) = \hbar\,\iota(\mathfrak{r}),$$

which leads to the following corollary of Proposition 2.5:

Corollary 2.8. $\Phi_\hbar(u) = \Phi_0(e^{\hbar\iota(\mathfrak{r})}u) = \Phi_0(u) + \hbar\,\Phi_0(\iota(\mathfrak{r})u) + \frac{\hbar^2}{2!}\Phi_0(\iota(\mathfrak{r})^2u) + \dots.$ □

The map $\Phi_\hbar$ is thus given as a deformation of the Chevalley projection Φ_0. We emphasise that $\Phi_\hbar$ is injective on the space J of invariants for $\hbar \neq 0$ while $\Phi_0 = \Phi_\hbar|_{\hbar=0}$ is not.

3. The algebras of invariants and Kostant's ρ-decomposition of $C\ell(\mathfrak{g})$

This section contains the information on the structure of the algebras $J_S = S(\mathfrak{g})^\mathfrak{g}$, $J = C\ell(\mathfrak{g})^\mathfrak{g}$ and $C\ell(\mathfrak{g})$ which will be used in the proof of our main results. We will recall several theorems of Kostant from [19] where it is assumed that $(\ ,\ )$ is the Killing form on $\mathfrak{g}$; however, they are easily seen to hold when $(\ ,\ )$ is any non-degenerate *ad*-invariant form.

3.1. **Generators of the algebra of symmetric invariants.** Recall from Section 1 that the Chevalley projection map establishes an algebra homomorphism between $J_S = S(\mathfrak{g})^\mathfrak{g}$ and the algebra $S(\mathfrak{h})^W$ of Weyl group invariants in the polynomial algebra $S(\mathfrak{h})$. From the invariant theory of reflection groups [24, 9] it follows that J_S has $r = \mathit{rank}(\mathfrak{g})$ algebraically independent homogeneous generators $f_1, f_2, \dots, f_r$. These are defined up to multiplication by non-zero constants and modulo $(J_S^+)^2$, where $J_S^+ = J_S \cap \oplus_{n>0} S^n(\mathfrak{g})$. We will denote by P_S the linear span of some chosen $f_1, f_2, \dots, f_r$. The space P_S is, in general, not uniquely defined and depends on the choice of the f_i.

Define the positive integers $m_1, \dots, m_r$ by $\deg f_i = m_i + 1$. The numbers m_i are independent (up to reordering) of the choice of a particular set of the f_i and are called the exponents of $\mathfrak{g}$.

3.2. **Primitive skew-symmetric invariants.** It turns out that there is a skew symmetric counterpart, P, of the space $P_S \subset J_S$. First of all, extend the *ad*-invariant form $(\ ,\ )$ from $\mathfrak{g}$ to the whole of $\bigwedge\mathfrak{g}$ in a standard way: $\bigwedge^m\mathfrak{g}$ is orthogonal to $\bigwedge^n\mathfrak{g}$ unless $m = n$, and

$$(x_1 \wedge \dots \wedge x_n, y_1 \wedge \dots \wedge y_n) = \det((x_i, y_j))_{i,j=1}^n$$

where $x_i, y_i \in \mathfrak{g}$. The space $P \subset \bigwedge\mathfrak{g}$ of primitive alternating invariants is defined as the $(\ ,\ )$-orthocomplement of $J^+ \wedge J^+$ in J^+, where J^+ is the augmentation ideal $J \cap (\sum_{m>0} \bigwedge^m\mathfrak{g})$. By a theorem of Koszul (see [19, Theorem 26]), the restriction of $(\ ,\ )$ to J and to J^+ is non-singular. Hence P is a graded subspace of J. Moreover, the dimension of P is equal to the rank r of $\mathfrak{g}$, and one actually knows the degrees where the graded components of P are located:

$$P = \mathrm{span}\,\{p_1, p_2, \dots, p_r\}, \qquad p_i \in (\textstyle\bigwedge^{2m_i+1}\mathfrak{g})^\mathfrak{g},$$

where m_i, as before, are the exponents of $\mathfrak{g}$.

Under a natural bijection between J and the homology $H_*(\mathfrak{g})$ of $\mathfrak{g}$, the elements of P correspond to what is known as primitive homology classes (see [19, 4.3]).

Now it turns out that the space J of invariants, with the product induced from $\bigwedge\mathfrak{g}$, is itself an exterior algebra. The Hopf-Koszul-Samelson theorem (see [19, 4.3] which refers to [21, Theorem 10.2]) asserts that

$$J = \textstyle\bigwedge P,$$

meaning that the map $\zeta\colon \bigwedge P \to J$, which is an algebra homomorphism extending the inclusion map $P \hookrightarrow J$ (the elements of P anticommute in J, being of odd degree), is an isomorphism.

3.3. **The subalgebra $J \subset C\ell(\mathfrak{g})$.** Even more surprising is the result, due to Kostant, that the subalgebra $J = C\ell(\mathfrak{g})^{\mathfrak{g}}$ of the Clifford algebra $C\ell(\mathfrak{g})$, is itself a Clifford algebra, generated by P as primitive generators.

Of course, the assertion $J = C\ell(P)$ has a chance to be valid only if a primitive tensor $p \in P$, being Clifford-squared, yields a constant. And this is indeed true; as shown in [19, Theorem B], primitive skew-symmetric invariants behave under Clifford multiplication as if they were elements of degree 1:

$$p \cdot p = (\alpha(p), p) \qquad \text{for } p \in P,$$

where α is defined as multiplication by the constant $(-1)^m$ on $P \cap \bigwedge^{2m+1}\mathfrak{g}$. A further result of Kostant [19, Theorem 35] asserts that the map $\zeta_{C\ell} \colon C\ell(P) \to J$, which extends, as an algebra homomorphism, the inclusion map $P \hookrightarrow J$, is an isomorphism of algebras. Here $C\ell(P)$ is the Clifford algebra of the space P equipped with the non-degenerate bilinear form $(p, q)_0 = (\alpha(p), q)$. For later use, we record this here as theorem.

Theorem 3.1 (Kostant). *In the above notation, the diagram*

$$\begin{array}{ccc} \bigwedge P & \xrightarrow[\sim]{\zeta} & J \\ {\scriptstyle\beta_{\wedge,P}}\big\downarrow & & \big\downarrow{\scriptstyle\beta_{\wedge,\mathfrak{g}}} \\ C\ell(P) & \xrightarrow[\sim]{\zeta_{C\ell}} & J \end{array}$$

commutes. Here $\beta_{\wedge,P} \colon \bigwedge P \to C\ell(P)$ and $\beta_{\wedge,\mathfrak{g}} \colon \bigwedge\mathfrak{g} \to C\ell(\mathfrak{g})$ are skew-symmetrisation maps for the respective exterior algebras.

3.4. **The Chevalley transgression map.** We briefly recall the useful transgression map, following [19, Section 6]. Let

$$d \colon \mathfrak{g} \to \bigwedge\nolimits^2 \mathfrak{g}, \qquad dx = \frac{1}{2}\sum e_a \wedge [e^a, x]$$

be the coboundary map for $\mathfrak{g}$. (Its extension to $\bigwedge\mathfrak{g}$ as a derivation of degree 1 is the coboundary in the standard Koszul complex for $\mathfrak{g}$.) Here $\{e_a\}$, $\{e^a\}$ is any pair of dual bases of $\mathfrak{g}$ with respect to the form $(\ ,\ )$. Now introduce the algebra homomorphism

$$s \colon S(\mathfrak{g}) \to \bigwedge\nolimits^{even}\mathfrak{g}, \qquad s(x_1 x_2 \dots x_n) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

between $S(\mathfrak{g})$ and the commutative subalgebra $\bigwedge^{even}\mathfrak{g} = \sum_n \bigwedge^{2n}\mathfrak{g}$ of $\bigwedge\mathfrak{g}$.

Denote by $\iota_S(x)f$ the directional derivative of $f \in S(\mathfrak{g})$ with respect to $x \in \mathfrak{g}$ (attention: $\mathfrak{g}$ is identified with its dual space $\mathfrak{g}^*$ via the form $(\ ,\ )$). In other words, $\iota_S(x)$ is the derivation of $S(\mathfrak{g})$ of degree -1 such that for y in $S^1(\mathfrak{g}) = \mathfrak{g}$, one has $\iota_S(x)y = (x, y) \in S^0(\mathfrak{g})$. The Chevalley transgression map may now be defined by the formula

$$t(f) = \frac{(m!)^2}{(2m+1)!}\sum_a e_a \wedge s(\iota_S(e^a)f)$$

due to Kostant [19, Theorem 64]. This maps symmetric tensors of degree $m+1$ to alternating tensors of degree $2m+1$. By a result of Chevalley (see [19, Theorem 66]), for any choice of the space P_S of primitive symmetric invariants, $t \colon P_S \to P$ is a linear isomorphism. (One observes that t vanishes on $(J_S^+)^2$.) One can choose $f_i \in P_S \cap S^{m_i+1}(\mathfrak{g})$ so that $t(f_i) = p_i$.

3.5. **The ρ-decomposition of $C\ell(\mathfrak{g})$.** Kostant's "separation of variables" result for the Clifford algebra [19] asserts that $C\ell(\mathfrak{g})$ is a free module over its subalgebra J. There is a subalgebra $E \subset C\ell(\mathfrak{g})$, which is in fact the Clifford centraliser of J in $C\ell(\mathfrak{g})$, so that the Clifford algebra factorises as

$$C\ell(\mathfrak{g}) = E \otimes J,$$

where $\otimes$ is realised by Clifford multiplication. Moreover, E can also be described as follows. For $x \in \mathfrak{g}$, denote

$$\delta(x) = \frac{1}{4}\sum_a e_a \cdot [e^a, x] \in C\ell(\mathfrak{g}),$$

where, as usual, $\{e_a\}$, $\{e^a\}$ are a pair of dual bases of $\mathfrak{g}$. It is easy to show that in fact, identifying the spaces $\bigwedge\mathfrak{g}$ and $C\ell(\mathfrak{g})$, one has $\delta(x) = \frac{1}{2}dx$ where $dx \in \bigwedge^2\mathfrak{g}$ is the coboundary of x introduced earlier. The equation $\delta([x,y]) = \delta(x)\delta(y) - \delta(y)\delta(x)$ holds in the Clifford algebra $C\ell(\mathfrak{g})$; see [19, Proposition 28]. Therefore, δ extends to a homomorphism

$$\delta\colon U(\mathfrak{g}) \to C\ell(\mathfrak{g})$$

of associative algebras. One has $E = \delta(U(\mathfrak{g}))$.

Now, let ρ be the half sum of positive roots of $\mathfrak{g}$, and let V_ρ denote the irreducible $\mathfrak{g}$-module with highest weight ρ. Kostant identifies E, as an algebra and a $\mathfrak{g}$-module, with the matrix algebra $\operatorname{End} V_\rho$, which is why the factorisation $C\ell(\mathfrak{g}) = E \otimes J$ is referred to as the ρ-decomposition of $C\ell(\mathfrak{g})$. Indeed, let $x_1, \dots, x_n$ be an ordering of positive root vectors in $\mathfrak{g}$, and let $\mu_+ = x_1 \cdot \dots \cdot x_n$ in $C\ell(\mathfrak{g})$. Let $z \in U(\mathfrak{g})$ act on the subspace $E\mu_+$ of $C\ell(\mathfrak{g})$ via left multiplication by $\delta(z)$. This makes $E\mu_+$ into a $U(\mathfrak{g})$-module isomorphic to V_ρ, with highest weight vector μ_+. The action of E on $V_\rho \cong E\mu_+$ by left multiplication induces a homomorphism $E \to \operatorname{End} V_\rho$. It turns out to be an isomorphism [19, Theorem 40].

4. Φ is an isomorphism between $C\ell(\mathfrak{g})^{\mathfrak{g}}$ and $C\ell(\mathfrak{h})$

4.1. **The first main theorem.** In this section we prove our first main result about the Clifford algebra analogue, $\Phi\colon C\ell(\mathfrak{g}) \to C\ell(\mathfrak{h})$, of the Harish-Chandra map. We will use the notation introduced in previous sections.

Theorem 4.1. *The restriction of the Harish-Chandra map Φ to the $\mathfrak{g}$-invariants in $C\ell(\mathfrak{g})$ is a superalgebra isomorphism $\Phi\colon C\ell(\mathfrak{g})^{\mathfrak{g}} \to C\ell(\mathfrak{h})$.*

Proof. We use Kostant's ρ-decomposition, $C\ell(\mathfrak{g}) = E \otimes J$, of the Clifford algebra $C\ell(\mathfrak{g})$. Let us denote by $E^{\mathfrak{h}}$ the subalgebra $\{u \in E \mid \theta(\mathfrak{h})u = 0\}$ of $C\ell(\mathfrak{g})^{\mathfrak{h}}$. Then we have the tensor factorisation

$$C\ell(\mathfrak{g})^{\mathfrak{h}} = E^{\mathfrak{h}} \otimes J.$$

Recall from the previous Section that E is the image of $U(\mathfrak{g})$ under the algebra map $\delta\colon U(\mathfrak{g}) \to C\ell(\mathfrak{g})$. As δ is also a $\mathfrak{g}$-module map, $E^{\mathfrak{h}} = \delta(U(\mathfrak{g})^{\mathfrak{h}})$, where $U(\mathfrak{g})^{\mathfrak{h}}$ is $ad\,\mathfrak{h}$-invariants in $U(\mathfrak{g})$. Choose a basis of $U(\mathfrak{g})^{\mathfrak{h}}$ consisting of monomials $y_{a_1}\dots y_{a_k}h_{b_1}\dots h_{b_l}x_{c_1}\dots x_{c_m}$, where y_a (resp. x_c) are negative (resp. positive) root vectors in $\mathfrak{g}$, h_b are vectors in the Cartan subalgebra, and the product is, of course, in $U(\mathfrak{g})$. To see what the δ-image of such a monomial can be, we use the following

Lemma 4.2. (*i*) *For any $h \in \mathfrak{h}$, $\Phi(\delta(h))$ is equal to the constant $\rho(h)$, where $\rho \in \mathfrak{h}^*$ is half the sum of positive roots of $\mathfrak{g}$.*

(*ii*) *For any $x \in \mathfrak{n}^+ \subset \mathfrak{g}$, $\delta(x)$ is in $C\ell(\mathfrak{g}) \cdot \mathfrak{n}^+$.*

Proof of the lemma. Choose a pair of dual bases of $\mathfrak{g}$ in the following special way. Let $x_1, \dots, x_n$ be positive root vectors in $\mathfrak{g}$, corresponding to the positive roots $\beta_1, \dots, \beta_n$ and constituting the basis of $\mathfrak{n}^+$. Let $y_1, \dots, y_n$ be negative root vectors so that $(x_i, y_i) = 1$, and let $h_1, \dots, h_r$ be some basis of $\mathfrak{h}$, orthonormal with respect to $(\, , \,)$. The bases $x_1, \dots, x_n; h_1, \dots, h_r; y_1, \dots, y_n$ and $y_1, \dots, y_n; h_1, \dots, h_r; x_1, \dots, x_n$ of $\mathfrak{g}$ are dual with respect to the *ad*-invariant non-degenerate form $(\, , \,)$. By definition of δ and using the fact that $\mathfrak{h}$ is an Abelian Lie subalgebra of $\mathfrak{g}$, one calculates

$$\delta(h) = \frac{1}{4}\sum_{i=1}^{n}(x_i \cdot [y_i, h] + y_i \cdot [x_i, h]) = \frac{1}{4}\sum_i \beta_i(h)(x_i y_i - y_i x_i).$$

Observe that $x_i y_i - y_i x_i = 2 - 2 y_i x_i$ in the Clifford algebra, and $\Phi(y_i x_i) = 0$ because, by definition of Φ, the kernel of Φ contains $\mathfrak{n}_- \cdot C\ell(\mathfrak{g})$ and $C\ell(\mathfrak{g}) \cdot \mathfrak{n}_+$. Thus one obtains $\Phi(\delta(h)) = \frac{1}{4}\sum_i \beta_i(h) \cdot 2 = \rho(h)$, establishing part (i) of the Lemma.

Now for $x \in \mathfrak{n}^+$, calculation of $\delta(x)$ with respect to the same special pair of dual bases of $\mathfrak{g}$ will yield an expression with the following terms: $x_i \cdot [y_i, x]$, $y_i \cdot [x_i, x]$ and $h_j \cdot [h_j, x]$. The latter two clearly belong to $C\ell(\mathfrak{g})\mathfrak{n}_+$. Rewrite $x_i \cdot [y_i, x]$ as $-[y_i, x] \cdot x_i + 2(x_i, [y_i, x])$. Here $-[y_i, x] \cdot x_i$ is again in $C\ell(\mathfrak{g})\mathfrak{n}_+$, and $(x_i, [y_i, x]) = ([x_i, y_i], x) = 0$ because $[x_i, y_i] \in \mathfrak{h}$ and $x \in \mathfrak{n}_+$. Thus $\delta(x) \in C\ell(\mathfrak{g}) \cdot \mathfrak{n}_+$. The Lemma is proved. $\square$

Remark 4.3. The proof of Lemma 4.2 is similar to [19, Proposition 37, Lemma 38 and Theorem 39]. These statements lead to a Clifford algebra realisation of the representation with highest weight ρ of a semisimple Lie algebra (Chevalley-Kostant construction). This construction can be generalised to central extensions of the corresponding loop algebra and in particular for $\widehat{\mathfrak{sl}_2}$. See the paper [16] by Joseph. One may realise the basic modules of $\widehat{\mathfrak{sl}_2}$, which is done by Greenstein and Joseph in [14] and has no analogue in the semisimple case. In the infinite dimensional case, the ordering of the factors in the expression for $\delta(h)$ becomes crucial.

We now continue the proof of Theorem 4.1. Consider a typical monomial $y_{a_1} \dots y_{a_k} \; h_{b_1} \dots h_{b_l} \; x_{c_1} \dots x_{c_m}$ in $U(\mathfrak{g})^{\mathfrak{h}}$ as above. If $m > 0$, the δ-image of this monomial lies in $C\ell(\mathfrak{g})\delta(x_{c_m})$, which is in $C\ell(\mathfrak{g})\mathfrak{n}^+$ by Lemma 4.2. By definition of Φ, $C\ell(\mathfrak{g})\mathfrak{n}^+$ lies in the kernel of Φ, thus the $\Phi \circ \delta$-image of the monomial is zero.

If $m = 0$, then $k = 0$ because the monomial must have weight zero with respect to the adjoint action of $\mathfrak{h}$ on $U(\mathfrak{g})$. The δ-image of the monomial $h_{b_1} \dots h_{b_l}$ is $\delta(h_{b_1}) \dots \delta(h_{b_l})$. By Lemma 4.2 one has $\Phi(\delta(h_{b_1}) \dots \delta(h_{b_l})) = \rho(h_{b_1}) \dots \rho(h_{b_l}) \in \mathbb{C}$. Thus, we have shown that

$$\Phi(E^{\mathfrak{h}}) = \mathbb{C}.$$

Because $\Phi \colon C\ell(\mathfrak{g})^{\mathfrak{h}} \to C\ell(\mathfrak{h})$ is a superalgebra homomorphism (by Lemma 2.4) and is surjective (coincides with the identity map on $C\ell(\mathfrak{h}) \subset C\ell(\mathfrak{g})^{\mathfrak{h}}$), we have $C\ell(\mathfrak{h}) = \Phi(E^h)\Phi(J)$. It follows that $C\ell(\mathfrak{h}) = \Phi(J)$, that is, $\Phi \colon J \to C\ell(\mathfrak{h})$ is surjective, hence bijective by comparison of dimensions (both are 2^r). Theorem 4.1 is proved. $\square$

4.2. **A formula for $\Phi_\hbar \circ \delta$.** Looking at the proofs of Lemma 4.2 and Theorem 4.1, we conclude that the calculations which have been made lead to a formula for the map $\Phi \circ \delta \colon U(\mathfrak{g}) \to \mathbb{C}$. Namely, it is apparent that $\Phi(\delta(u)) = \Psi(u)(\rho)$, where $\Psi \colon U(\mathfrak{g}) \mapsto S(\mathfrak{h})$ is the classical Harish-Chandra map introduced in 1.2, and $\cdot \, (\rho)$ denotes the evaluation of an element of $S(\mathfrak{h})$, viewed as a polynomial function on the space $\mathfrak{h}^*$, at the point $\rho \in \mathfrak{h}^*$. For later purposes we will need a slightly more general version of this formula, which is nevertheless established by a completely analogous calculation. Recall the "deformed" Harish-Chandra map $\Phi_\hbar \colon C\ell(\mathfrak{g}) \to C\ell(\mathfrak{h})$, introduced in Remark 2.7.

Lemma 4.4. *The map $\Phi_\hbar \circ \delta \colon U(\mathfrak{g}) \to \mathbb{C}$ is given by*

$$\Phi_\hbar(\delta(u)) = \Psi(u)(\hbar\rho). \quad \square$$

4.3. **Φ identifies P and $\mathfrak{h}$.** In the proof of Theorem 4.1 we relied on the result, due to Kostant, that $\mathit{C\ell}(\mathfrak{g})$ factorises as $E \otimes J$. We are going to obtain more information about the Harish-Chandra map $\Phi\colon \mathit{C\ell}(\mathfrak{g})^{\mathfrak{g}} \to \mathit{C\ell}(\mathfrak{h})$ using Kostant's description of J as the Clifford algebra $\mathit{C\ell}(P)$, where P is the space of primitive $\mathfrak{g}$-invariants in $\mathit{C\ell}(\mathfrak{g})$ as in the previous Section. It is one of the key results of [19] that

$$\iota(x)p \in E \qquad \text{for } p \in P, \quad x \in \mathfrak{g};$$

see [19, Theorem E]. From this, we deduce our

Proposition 4.5. *The restriction of the Harish-Chandra map Φ to the space $P \subset J$ of primitive invariants is a bijective linear map between P and the Cartan subalgebra $\mathfrak{h}$.*

Proof. Take a primitive invariant $p \in P$. For any $h \in \mathfrak{h}$, one has $\iota(h)p \in E$ by the above result of Kostant. Therefore, it follows from Lemma 4.2 that $\Phi(\iota(h)p) \in \mathbb{C}$. More precisely, $\Phi(\iota(h)p)$ is in the one-dimensional subspace $\mathbb{C} \cdot 1 = \bigwedge^0 \mathfrak{h}$ of $\bigwedge \mathfrak{h}$. Let us now use

Lemma 4.6. *For any $h \in \mathfrak{h}$ and $u \in \mathit{C\ell}(\mathfrak{g})^{\mathfrak{h}}$, one has $\iota(h)\Phi(u) = \Phi(\iota(h)u)$.*

Proof of Lemma 4.6. We have already observed that $\mathit{C\ell}(\mathfrak{g})^{\mathfrak{h}}$ decomposes as a direct sum $\mathit{C\ell}(\mathfrak{h}) \oplus \mathit{C\ell}(\mathfrak{g})\mathfrak{n}^+ \cap \mathit{C\ell}(\mathfrak{g})^{\mathfrak{h}}$. Because $\iota(h)$ is a superderivation of $\mathit{C\ell}(\mathfrak{g})$ (Lemma 2.3) and $\iota(h)\mathfrak{n}^+ = 0$, $\iota(h)$ preserves the subspaces $\mathit{C\ell}(\mathfrak{h})$ and $\mathit{C\ell}(\mathfrak{g})\mathfrak{n}^+ \cap \mathit{C\ell}(\mathfrak{g})^{\mathfrak{h}}$ of $\mathit{C\ell}(\mathfrak{g})$, hence commutes with the projection onto $\mathit{C\ell}(\mathfrak{h})$. $\square$

Let $\Phi(p)$ be some element $q \in \bigwedge \mathfrak{h}$. Applying Lemma 4.6 we establish that $\iota(h)q \in \bigwedge^0 \mathfrak{h}$ for all $h \in \mathfrak{h}$. The restriction of the form $(\ ,\ )$ to $\mathfrak{h}$ is non-degenerate, therefore there exists $h^1 \in \mathfrak{h} = \bigwedge^1 \mathfrak{h}$ such that $\iota(h)h^1 = \iota(h)q$ for all $h \in \mathfrak{h}$. The intersection of kernels of all contraction operators $\iota(\mathfrak{h})$ in the exterior algebra $\bigwedge \mathfrak{h}$ is its zero degree part, $\bigwedge^0 \mathfrak{h} = \mathbb{C}$: this fact can be checked directly and is a particular case of a Nichols algebra property (see Remark 2.1 above and [4, Criterion 3.2]). Thus, $q = h^1 + h^0$ for some $h^0 \in \bigwedge^0 \mathfrak{h}$.

But Φ is a map of superspaces by Lemma 2.4, and a primitive invariant $p \in P$ is odd. Therefore, the even component h^0 of q is zero, thus $\Phi(p) \in \mathfrak{h}$. Hence $\Phi(P) \subset \mathfrak{h}$, and by injectivity of Φ on $\mathit{C\ell}(\mathfrak{g})^{\mathfrak{g}}$ (Theorem 4.1) and comparison of dimensions, $\Phi(P) = \mathfrak{h}$. Proposition 4.5 is proved. $\square$

4.4. **Φ induces an isomorphism between J and $\bigwedge \mathfrak{h}$.** We finish this Section with an observation that the Harish-Chandra map Φ respects not only Clifford but also exterior multiplication on the space of $\mathfrak{g}$-invariants in $\bigwedge \mathfrak{g}$.

Corollary 4.7. *The map $\Phi\colon (\bigwedge \mathfrak{g})^{\mathfrak{g}} \to \bigwedge \mathfrak{h}$ is an algebra isomorphism.*

Remark 4.8. The Corollary asserts that the restriction Φ to J respects the wedge multiplication. This is not trivial, because it is not readily seen from the construction of Φ why $\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)$ when $a, b \in J$. Recall that in terms of the wedge product, the map Φ is given by the r-matrix formula (Proposition 2.5), and note that in general, $\Phi(a \wedge b) \neq \Phi(a) \wedge \Phi(b)$ for $a, b \in (\bigwedge \mathfrak{g})^{\mathfrak{h}}$.

Proof of Corollary 4.7. We need to show that the wedge product in $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is respected by the map, fully written as $\sigma_{\mathfrak{h}} \circ \Phi \circ \sigma_{\mathfrak{g}}^{-1}$, where $\sigma_{\mathfrak{h}}\colon \mathit{C\ell}(\mathfrak{h}) \to \bigwedge \mathfrak{h}$ and $\sigma_{\mathfrak{g}}\colon \mathit{C\ell}(\mathfrak{g}) \to \bigwedge \mathfrak{g}$ are maps identifying the underlying linear space of a Clifford algebra with the corresponding exterior algebra; see 2.1. But

by Theorem 3.1 it is enough to show that $\sigma_{\mathfrak{h}} \circ \Phi \circ \sigma_P^{-1}$ respects the wedge product in $\bigwedge P$, where P is the space of primitive invariants in $\bigwedge \mathfrak{g}$. Let $p_1, \ldots, p_r$ be a basis of P such that p_i are homogeneous in $\bigwedge \mathfrak{g}$ and are pairwise orthogonal with respect to the form $(\ ,\ )$ extended to $\bigwedge \mathfrak{g}$. Then p_i are also orthogonal with respect to the form $(\alpha(\cdot), \cdot)$, which gives rise to the Clifford algebra $C\ell(P)$ as in 3.3. It follows that the p_i pairwise anticommute in $C\ell(P)$, hence also in $C\ell(\mathfrak{g})$.

For each subset $I = \{i_1 < \cdots < i_k\}$ of $\{1, \ldots, r\}$, denote $p_I = p_{i_1} \wedge \cdots \wedge p_{i_k}$. The 2^r elements p_I form a basis of $\bigwedge P$. By orthogonality of the p_i, $p_I = \sigma_P(p_{i_1} p_{i_2} \ldots p_{i_k})$; applying Φ gives $h_{i_1} h_{i_2} \ldots h_{i_k} \in C\ell(\mathfrak{h})$, where $h_i = \Phi(p_i)$. By Proposition 4.5, $h_1, \ldots, h_r$ form a basis of the Cartan subalgebra $\mathfrak{h}$. Moreover, this basis is orthogonal with respect to the restriction of the form $(\ ,\ )$ on $\mathfrak{h}$, because the h_i pairwise anticommute in the Clifford algebra $C\ell(\mathfrak{h})$ being the images of the p_i. Therefore, $\sigma_{\mathfrak{h}}(h_{i_1} h_{i_2} \ldots h_{i_k}) = h_{i_1} \wedge h_{i_2} \wedge \cdots \wedge h_{i_k}$, which may be denoted by h_I.

Thus, modulo the appropriate identifications, the map $\Phi \colon (\bigwedge \mathfrak{g})^{\mathfrak{g}} \to \bigwedge \mathfrak{h}$ is given on the basis $\{p_I \mid I \subseteq \{1, \ldots, r\}\}$ of $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ by $\Phi(p_I) = h_I$. It manifestly follows that Φ is an isomorphism of exterior algebras. $\square$

5. The principal basis of the Cartan subalgebra

In the previous Section we established that the Harish-Chandra map $\Phi \colon C\ell(\mathfrak{g}) \to C\ell(\mathfrak{h})$ restricts to a bijective linear map between the space P of primitive $\mathfrak{g}$-invariants in $C\ell(\mathfrak{g})$ and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Recall that P is a graded subspace of the exterior algebra $\bigwedge \mathfrak{g}$ of $\mathfrak{g}$. The linear bijection $\Phi \colon P \to \mathfrak{h}$ thus induces a grading on the Cartan subalgebra $\mathfrak{h}$. We will now show that this grading coincides with one arising in a different context — from the decomposition of $\mathfrak{g}^\vee$, the Langlands dual Lie algebra of $\mathfrak{g}$, as a module over its principal three-dimensional simple subalgebra.

5.1. **Principal three-dimensional simple subalgebras.** In this subsection, we briefly recall known facts about principal three-dimensional simple subalgebras (principal TDS) of $\mathfrak{g}$. Our standard reference for this is [17, §5].

Let r be the rank of a semisimple Lie algebra $\mathfrak{g}$. A nilpotent element e of $\mathfrak{g}$ is called principal nilpotent (or regular nilpotent), if the centraliser $\mathfrak{g}^e$ of e in $\mathfrak{g}$ is of minimal possible dimension, namely $\dim \mathfrak{g}^e = r$. By the well-known Jacobson-Morozov theorem, any non-zero nilpotent element $e \in \mathfrak{g}$ can be included in an $\mathfrak{sl}_2$-triple (e, h, f) of elements of $\mathfrak{g}$, i.e., a triple that fulfils the relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Such an $\mathfrak{sl}_2$-triple is called principal, if e is principal nilpotent. The linear span of a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is referred to as a principal three-dimensional simple subalgebra (principal TDS) of $\mathfrak{g}$.

The following characterisation of principal TDS is due to Kostant:

Lemma 5.1. *If $\mathfrak{a}$ is a principal TDS of $\mathfrak{g}$, then $\mathfrak{g}$, viewed as an $\mathfrak{a}$-module via the adjoint action, is a direct sum of precisely r simple $\mathfrak{a}$-modules.* $\square$

If a subalgebra $\mathfrak{a} \cong \mathfrak{sl}_2$ of $\mathfrak{g}$ is not principal, then $\mathfrak{g}$ is a direct sum of strictly more than r simple $\mathfrak{a}$-modules.

All principal TDS $\mathfrak{g}$ are conjugate (with respect to the action of the adjoint group of $\mathfrak{g}$). We will be interested in one particular principal TDS $\mathfrak{a}_0$ of $\mathfrak{g}$, discovered independently by Dynkin and de Siebenthal. To define $\mathfrak{a}_0$, fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and recall that our chosen ad-invariant form $(\ ,\ )$, restricted to $\mathfrak{h}$, is non-degenerate, hence one may identify $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$. In particular, the root system of $\mathfrak{g}$ becomes a subset of $\mathfrak{h}$; a root α and the corresponding coroot $\alpha^\vee$ are related via $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Choose a set of simple roots $\alpha_1, \dots, \alpha_r \in \mathfrak{h}$. Denote by $\rho^\vee$ the element of $\mathfrak{h}$ defined by the condition $(\alpha_i, \rho^\vee) = 1$, $i = 1, \dots, r$; this element is half the sum of positive coroots of $\mathfrak{g}$. Let e_i be root vectors corresponding to the roots α_i, and f_i be root vectors corresponding to roots $-\alpha_i$ normalised so that $(e_i, f_i) = 1$. Observe that $[e_i, f_i] = \alpha_i \in \mathfrak{h}$. Indeed, for arbitrary $h \in \mathfrak{h}$ one has $([e_i, f_i], h) = (e_i, [f_i, h]) = (e_i, \alpha_i(h) f_i) = (\alpha_i, h)$. Put

$$e_0 = \sum_{i=1}^r e_i, \qquad h_0 = 2\rho^\vee, \qquad f_0 = \sum_{i=1}^r c_i f_i,$$

where c_i are the coefficients in the expansion $2\rho^\vee = \sum_{i=1}^r c_i \alpha_i$. It is not difficult to show that (e_0, h_0, f_0) is an $\mathfrak{sl}_2$-triple (using the fact that $(\alpha_i, 2\rho^\vee) = 2$ for all i, that $[e_i, f_j] = 0$ for $i \neq j$, and that $[e_i, f_i] = \alpha_i$). Moreover, e_0 is a principal nilpotent element of $\mathfrak{g}$. The principal TDS $\mathfrak{a}_0$ of $\mathfrak{g}$ is the linear span of the triple (e_0, h_0, f_0).

5.2. **The principal grading on $\mathfrak{h}$.** Recall that under the adjoint action of a principal TDS $\mathfrak{a}$, the Lie algebra $\mathfrak{g}$ breaks down into a direct sum of $r = \mathit{rank}\, \mathfrak{g}$ simple $\mathfrak{a}$-modules. The dimensions of these simple modules were determined by Kostant in [17]: writing V_d for the d-dimensional $\mathfrak{sl}_2$-module, one has

$$\mathfrak{g} \cong V_{2m_1+1} \oplus \dots \oplus V_{2m_r+1},$$

where m_i, $i = 1, 2, \dots, r$, are the exponents of $\mathfrak{g}$; see 3.1. Each $\mathfrak{sl}_2$-module V_{2m_i+1}, being of odd dimension, has a one-dimensional zero weight subspace $V^0_{2m_i+1}$. We now turn to the case $\mathfrak{a} = \mathfrak{a}_0$, the distinguished principal TDS introduced above. Fix an isomorphism between $\mathfrak{g}$ and $\oplus_i V_{2m_i+1}$ and regard each V_{2m_i+1} as an $\mathfrak{a}_0$-submodule of $\mathfrak{g}$. Clearly, $\oplus_i V^0_{2m_i+1}$ is the centraliser, $\mathfrak{g}^{h_0}$, of $h_0 = 2\rho^\vee$ in $\mathfrak{g}$. Since all root vectors in $\mathfrak{g}$ are eigenvectors of $ad\, h_0$ corresponding to non-zero eigenvalues, it follows that $\mathfrak{g}^{h_0} = \mathfrak{h}$, the Cartan subalgebra of $\mathfrak{g}$ (i.e., h_0 is a regular semisimple element of $\mathfrak{g}$). One has the direct sum decomposition

$$\mathfrak{h} = V^0_{2m_1+1} \oplus \dots \oplus V^0_{2m_r+1}.$$

This decomposition of $\mathfrak{h}$ is not canonical, because there is some freedom in choosing the isomorphism between $\mathfrak{g}$ and $\oplus_i V_{2m_i+1}$. However, for each d, the V_d-primary component $\oplus\{V_{2m_i+1} : 2m_i + 1 = d\}$ is canonically defined. There is thus a grading

$$\mathfrak{h} = \oplus_d \mathfrak{h}_d, \qquad \mathfrak{h}_d = \mathfrak{h} \cap \oplus\{V_{2m_i+1} : 2m_i + 1 = d\}$$

on the Cartan subalgebra $\mathfrak{h}$, which only depends on the choice of a set of simple roots of $\mathfrak{h}$ in $\mathfrak{g}$ (i.e., the choice of a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$). Note that the element $\rho^\vee \in \mathfrak{h}$ and the grading do not depend on a particular non-degenerate ad-invariant form $(\ ,\ )$. We refer to this grading as the principal grading on $\mathfrak{h}$.

Remark 5.2. There is another grading on $\mathfrak{g}$ associated to any $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, namely the eigenspace decomposition of $ad\, h$. Such gradings are referred to as Dynkin gradings in [12]. Note that the Dynkin grading arising from (e_0, h_0, f_0) has $\mathfrak{h}$ as the degree zero subspace and breaks down each of the V_{2m_i+1} into a direct sum of 1-dimensional graded subspaces, so it is, in a sense, transversal to the principal grading.

5.3. **Elementary properties of the principal grading.** Let us observe a few straightforward properties of the principal grading on $\mathfrak{h}$.

Lemma 5.3. *Let $\mathfrak{h} = \oplus_d \mathfrak{h}_d$ be the principal grading on the Cartan subalgebra $\mathfrak{h}$.*

(a) *Non-zero homogeneous components $\mathfrak{h}_d$ occur only in degrees $d = 2m+1$, where m is an exponent of $\mathfrak{g}$. The least such degree is 3. The dimension of $\mathfrak{h}_{2m+1}$ is the number of times m occurs as an exponent of $\mathfrak{g}$.*

(b) $\dim \mathfrak{h}_3$ *is the number of connected components in the Dynkin diagram of $\mathfrak{g}$.*

(c) *The element $\rho^\vee$ of $\mathfrak{h}$ belongs to $\mathfrak{h}_3$.*

(d) *Homogeneous components of $\mathfrak{h}$ of different degrees are orthogonal with respect to any ad-invariant form (,) on $\mathfrak{g}$.*

Proof. (a) is apparent from the definition of the principal grading. To establish (b), one observes that 1 occurs as an exponent of $\mathfrak{g}$ as many times as is the number of summands in the decomposition of $\mathfrak{g}$ into a direct sum of simple Lie algebras, which is the number of connected components of the Dynkin diagram of $\mathfrak{g}$. (A simple Lie algebra has only one exponent equal to 1, and taking the direct sum of semisimple Lie algebras corresponds to taking the union of the multisets of exponents.) Note that $\mathfrak{a}_0 \subset \mathfrak{g}$ is a three-dimensional $\mathfrak{a}_0$-submodule of $\mathfrak{g}$, hence $\mathfrak{a}_0 \cap \mathfrak{h} = \mathbb{C}\rho^\vee \subset \mathfrak{h}_3$ and (c) follows. To prove (d), take $h, k \in \mathfrak{h}$ such that $h \in V^0_{2m+1} \subset \mathfrak{g}$ and $k \in V^0_{2n+1} \subset \mathfrak{g}$, with $m < n$ two distinct exponents of $\mathfrak{g}$. By the representation theory of $\mathfrak{sl}_2$, there is a highest weight vector $x \in V_{2n+1}$ for $\mathfrak{a}_0$, such that $k = (ad\, f_0)^n x$. Then $(h,k) = ((ad\, f_0)^n x, k) = (-1)^n (x, (ad\, f_0)^n k)$. But $(ad\, f_0)^n k = 0$, thus h, k are orthogonal. $\square$

5.4. **The principal basis of $\mathfrak{h}$ in a simple Lie algebra $\mathfrak{g}$.** When $\mathfrak{g}$ is simple, we can say more about the principal grading on $\mathfrak{h}$.

Lemma 5.4. *Let $\mathfrak{h}$ be the Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is not of type D_r with r even, then all non-zero homogeneous components of the principal grading $\mathfrak{h} = \oplus_d \mathfrak{h}_d$ are one-dimensional. If $\mathfrak{g}$ is of type D_r with r even, then* $\dim \mathfrak{h}_d = 1$ *for $d \neq r-1$, and* $\dim \mathfrak{h}_{r-1} = 2$.

Proof. According to [6, Table IV], simple Lie algebras of types other than D_r with even r have distinct exponents; in type D_r with even r all exponents have multiplicity 1 except $r/2 - 1$ which has multiplicity 2. Apply Lemma 5.3(a). $\square$

Thus, in all simple Lie algebras apart from $\mathfrak{so}(2r)$, r even, a choice, $\mathfrak{b} \supset \mathfrak{h}$, of a Borel and a Cartan subalgebra gives rise to a distinguished basis $h_1, h_2, \ldots, h_r$ of $\mathfrak{h}$. The vector h_i is determined, up to a scalar factor, by the condition $\mathfrak{h}_{2m_i+1} = \mathbb{C}h_i$. This basis is orthogonal with respect to the Killing form, and $h_1 = \rho^\vee$. We call $h_1, \ldots, h_r$ the principal basis of $\mathfrak{h}$.

For $\mathfrak{g} \cong \mathfrak{so}(2r)$, $r-2$ vectors of the principal basis are determined up to a scalar factor, and there is freedom in choosing the remaining pair of vectors which must span a given two-dimensional subspace $\mathfrak{h}_{r-1}$ of $\mathfrak{h}$. We still call any basis, obtained by this procedure, a principal basis of $\mathfrak{h}$.

5.5. **The grading on $\mathfrak{h}$ induced from $\mathfrak{g}^\vee$.** Recall that we have identified the Cartan subalgebra $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$ via the non-degenerate *ad*-invariant form (,), so that both the roots and the coroots of $\mathfrak{g}$ lie in $\mathfrak{h}$. Consider $\mathfrak{g}^\vee$, the Langlands dual Lie algebra to $\mathfrak{g}$; that is, the complex semisimple Lie algebra with a root system dual to that of $\mathfrak{g}$. The roots of $\mathfrak{g}^\vee$ are the coroots of $\mathfrak{g}$, so we will assume that $\mathfrak{g}^\vee$ and $\mathfrak{g}$ share the same Cartan subalgebra $\mathfrak{h}^\vee = \mathfrak{h}$.

The above definition of principal grading applies to $\mathfrak{g}^\vee$. One has, therefore, a grading

$$\mathfrak{h} = \oplus_d \mathfrak{h}_d^\vee$$

on the Cartan subalgebra, which is the principal grading induced from $\mathfrak{g}^\vee$. This grading depends on a choice of simple coroots $\alpha_1^\vee, \dots, \alpha_r^\vee$ which is the same as a choice $\alpha_1, \dots, \alpha_r$ of simple roots of $\mathfrak{g}$.

If $\mathfrak{g}$ is a simple Lie algebra, $\mathfrak{g}^\vee$ is also simple, so in that case $\mathfrak{h}$ has a dual principal basis induced from $\mathfrak{g}^\vee$.

5.6. **Main result.** We are now ready to state and prove the final main result of the paper, Theorem 5.5 which also solves a conjecture of Kostant. Once again, recall that $P \subset (\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is the space of primitive skew-symmetric invariants. It is graded by degree in $\bigwedge \mathfrak{g}$. The space P is also viewed as a subspace of $C\ell(\mathfrak{g})^{\mathfrak{g}}$. We already know that under the Harish-Chandra map $\Phi\colon C\ell(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} C\ell(\mathfrak{h})$, the space P is isomorphically mapped onto $\mathfrak{h}$.

Theorem 5.5. *The grading on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, induced from the grading on P by degree via the map $\Phi\colon P \xrightarrow{\sim} \mathfrak{h}$, coincides with the principal grading $\mathfrak{h} = \oplus_d \mathfrak{h}_d^\vee$ on $\mathfrak{h}$ in the Langlands dual Lie algebra $\mathfrak{g}^\vee$.*

Proof. We are going to use a result due to Chevalley, cited in 3.4, that the primitive skew-symmetric invariants are transgressive; i.e., if $f_1, \dots, f_r$ are independent homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$, then $t(f_1), \dots, t(f_r)$ is a basis of P. Let $m_1 \le m_2 \le \cdots \le m_r$ be the exponents of $\mathfrak{g}$ so that $f_i \in S^{m_i+1}(\mathfrak{g})$. Denote $p_i = t(f_i)$, thus $p_i \in \bigwedge^{2m_i+1}\mathfrak{g}$. Furthermore, let $z_1, \dots, z_r$ be any basis of the Cartan subalgebra $\mathfrak{h}$ orthonormal with respect to the non-degenerate ad-invariant form $(\,,\,)$. As $\Phi(p_i)$ is an element of $\mathfrak{h}$ by Proposition 4.5, we may write

$$\begin{aligned}\Phi(p_i) &= \sum_{j=1}^r (\Phi(p_i), z_j) z_j = \sum_{j=1}^r \iota(z_j)\Phi(p_i)\cdot z_j = \sum_{j=1}^r \Phi(\iota(z_j)p_i)\cdot z_j \\ &= \sum_{j=1}^r \Phi(\iota(z_j)t(f_i))\cdot z_j,\end{aligned}$$

where we used Lemma 4.6 to exchange the maps Φ and $\iota(z_j)$. We now observe that further to being two examples of braided derivations (remark 2.1), the operators $\iota_S(z)\colon S^{m+1}(\mathfrak{g}) \to S^m(\mathfrak{g})$ and $\iota(z)\colon \bigwedge^{2m+1}\mathfrak{g} \to \bigwedge^{2m}\mathfrak{g}$ are intertwined (up to a scalar factor) by the Chevalley transgression map t. One has

$$\iota(z)t(f) = \frac{(m!)^2}{(2m)!} s(\iota_S(z)f) \qquad \text{for all } f \in J_S \cap S^{m+1}(\mathfrak{g}),\ z \in \mathfrak{g}$$

by [19, Theorem 73], where $s\colon S(\mathfrak{g}) \to \bigwedge \mathfrak{g}$ is the homomorphism introduced in 3.4. Hence

$$\Phi(p_i) = c \sum_{j=1}^r \Phi(s(\iota_S(z_j)f_i))\cdot z_j$$

for some non-zero constant c.

In the next calculation, we are going to use results from [19] which are obtained for the case of simple $\mathfrak{g}$. We thus assume $\mathfrak{g}$ to be simple until further notice.

Let us now assume that the independent homogeneous generators $f_1, \dots, f_r$ of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ of symmetric invariants are chosen in a particular way, so as to span the orthogonal complement to $(J_S^+)^2$ in J_S^+ (compare with 3.1). Here orthogonality is with respect to a natural extension of

the form $(\ ,\ )$ from $\mathfrak{g}$ onto $S(\mathfrak{g})$: identify $S(\mathfrak{g})$ with $S(\mathfrak{g}^*)$ which is acting on $S(\mathfrak{g})$ by differential operators with constant coefficients, i.e., to $f \in S(\mathfrak{g})$ there corresponds $\partial_f \in S(\mathfrak{g}^*)$; now put (f, g) to be the evaluation of the polynomial function $\partial_f g$ at zero. Such choice of primitive symmetric invariants is due to Dynkin. Denote by P_D the span of $f_1, \ldots, f_r$ where $\deg f_i = m_i + 1$, and let $(P_D)_{(m)} = P_D \cap S^{\le m}(\mathfrak{g})$.

The space P_D is especially useful for us because of the following property [19, Theorem 87]: there exist $u_1, \ldots, u_r \in P_D$, such that $u_i - f_i \in (P_D)_{(m_i)}$ and

$$\delta_{u_i}(z) = \frac{1}{2^{m_i}} s(\iota_S(z) f_i)$$

for any $z \in \mathfrak{g}$. Here

$$\delta_u \colon \mathfrak{g} \to C\ell(\mathfrak{g}), \qquad \delta_u(z) = \delta(\beta(\iota_S(z)u))$$

is a $\mathfrak{g}$-equivariant map from $\mathfrak{g}$ to $C\ell(\mathfrak{g})$, defined in [19, 6.12]; we remind the reader that $\beta \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is the PBW symmetrisation map. (It is clear from the definition of the map δ_u that the image of δ_u is in the subalgebra $E = \operatorname{Im}\delta$ of $C\ell(\mathfrak{g})$.) In fact, the u_i are not homogeneous, and are of the form $u_i = f_i + \sum_{k<i} a_{ik} f_k$ for some coefficients $a_{ik} \in \mathbb{C}$.

We substitute this in the previous formula for $\Phi(p_i)$ to obtain

$$\Phi(p_i) = c \sum_{j=1}^{r} \Phi(\delta_{u_i}(z_j)) z_j$$

for some non-zero c. The expression for the scalar $\Phi(\delta_{u_i}(z_j))$ is given in Lemma 4.4 (applied for $\hbar = 1$):

$$\Phi(\delta_{u_i}(z_j)) = \Psi(\beta(\iota_S(z_j)u_i))(\rho),$$

where $\Psi \colon U(\mathfrak{g}) \to S(\mathfrak{h})$ is the Harish-Chandra map, and $\ \cdot\ (\rho)$ means the evaluation of an element of $S(\mathfrak{h})$ at the point ρ. Observe that $\iota_S(z_j)u_i$ is a non-homogeneous element of $S^{\le m_i}(\mathfrak{g})$, with top degree homogeneous component equal to $\iota_S(z_j)f_i$. Moreover, $\Psi(\beta(\iota_S(z_j)u_i))$ is an element of $S^{\le m_i}(\mathfrak{h})$, also in general non-homogeneous, with component of top degree m_i equal to $\Psi_0(\iota_S(z_j)f_i)$, where $\Psi_0 \colon S(\mathfrak{g}) \to S(\mathfrak{h})$ is the Chevalley projection map.

We would like to obtain a more satisfactory expression for $\Phi(\delta_{u_i}(z_j))$. To do this, we introduce the deformation parameter $\hbar$ into the picture. It is easy to run the above argument for the Harish-Chandra map $\Phi_\hbar$ instead of Φ, obtaining

$$\Phi_\hbar(p_i) = c \sum_{j=1}^{r} \Phi_\hbar(\delta_{u_i}(z_j)) z_j.$$

We can now apply Lemma 4.4 for general $\hbar \in \mathbb{C}$, obtaining $\Phi_\hbar(\delta_{u_i}(z_j)) = \Psi(\beta(\iota_S(z_j)u_i))(\hbar\,\rho)$, which leads to

$$\Phi_\hbar(p_i) = c \sum_{j=1}^{r} \Psi(\beta(\iota_S(z_j)u_i))(\hbar\,\rho) z_j.$$

Now let us observe, crucially, that the map $\Phi_\hbar$ is given by its expansion in terms of $\hbar$ in Corollary 2.8:

$$\Phi_\hbar(p_i) = \sum_{s\ge 0} \hbar^s \Phi_0(\iota(\mathfrak{r})^s p_i) \qquad \in \qquad \sum_{s\ge 0} \hbar^s {\textstyle\bigwedge}^{2m_i+1-2s}\mathfrak{h}.$$

We know that $\Phi_\hbar(p_i)$ is an element of degree 1, which means that only the term containing $\hbar^{m_i}$ is non-zero. It follows that $\Phi_\hbar(p_i)$ is homogeneous in $\hbar$ of degree m_i. But this means that all terms

of degree lower than m_i in the $\Psi(\beta(\iota_S(z_j)u_i))(\hbar\,\rho)$ can be dropped, as they make no contribution. What remains is therefore

$$\Phi_\hbar(p_i) = c\sum_{j=1}^{r}\Psi_0(\iota_S(z_j)f_i)(\hbar\,\rho)z_j = c\sum_{j=1}^{r}(\iota_S(z_j)f_i)(\hbar\rho)z_j$$

(the Chevalley projection Ψ_0 is irrelevant, since we are evaluating at a point in $\mathfrak{h}^*$ anyway). Now we can put $\hbar = 1$ and rewrite the formula in the following very simple way:

$$\Phi(p_i) = c\,\iota_S(\rho)^{m_i} f_i.$$

Recall that this only holds when f_i is a homogeneous element of degree $m_i + 1$ in the Dynkin space P_D of primitive symmetric invariants. The formula may no longer be true if P_D is replaced by some other space P_S spanned by independent homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$.

The proof of the Theorem for simple Lie algebras is concluded with the use of Lemma 5.6 below, which is essentially analogous to the calculation carried out in [23] but is included here for the sake of completeness. To extend the result to semisimple Lie algebras, note that if $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l$ is a direct sum of pairwise commuting simple Lie algebras, then $\mathfrak{g}^\vee \cong \mathfrak{g}_1^\vee \oplus \ldots \oplus \mathfrak{g}_l^\vee$. The Cartan subalgebra $\mathfrak{h}^\vee$ of $\mathfrak{g}^\vee$ is the direct sum of Cartan subalgebras of the $\mathfrak{g}_i^\vee$, and other obvious choices made for the root system of $\mathfrak{g}^\vee$ ensure that the de Siebenthal-Dynkin canonical principal TDS of $\mathfrak{g}^\vee$ is the direct sum of principal TDS of each of $\mathfrak{g}_i^\vee$. Thus, the dth degree of the principal grading in $\mathfrak{h}^\vee$ is spanned by dth vectors of principal bases of the $\mathfrak{g}_i^\vee$. On the other hand, the Clifford algebra $C\ell(\mathfrak{g})$ is a tensor product of pairwise supercommuting Clifford algebras of the $\mathfrak{g}_i$. All this ensures that the primitive $\mathfrak{g}$-invariants P in $C\ell(\mathfrak{g})$ are a direct sum of spaces of primitive $\mathfrak{g}_i$-invariants in $C\ell(\mathfrak{g}_i)$, the homogeneous basis of each of which projects, under the Harish-Chandra map, to the corresponding Langlands dual principal basis in the Cartan subalgebra $\mathfrak{h}_i$. The result for semisimple algebras thus follows by a straightforward direct sum argument. □

Lemma 5.6. *Let $b_1, \ldots, b_r$ be algebraically independent homogeneous generators of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ of symmetric invariants of a simple Lie algebra $\mathfrak{g}$ such that $\deg b_k = m_k + 1$. Put $h_k = \iota_S(\rho)^{m_k} b_k$. If the h_k are non-zero and pairwise orthogonal with respect to the Killing form, then $h_1 \ldots, h_r$ are a principal basis of $\mathfrak{h}$ induced from the Langlands dual Lie algebra $\mathfrak{g}^\vee$.*

Proof. Let $W \subset GL(\mathfrak{h})$ be the Weyl group of $\mathfrak{g}$, which is the same as the Weyl group of $\mathfrak{g}^\vee$. The Langlands dual principal basis of $\mathfrak{h}$ is orthogonal with respect to the Killing form on $\mathfrak{g}^\vee$, which in restriction to $\mathfrak{h}$ is the unique, up to a scalar factor, W-invariant form on $\mathfrak{h}$. Thus, the Langlands dual principal basis is orthogonal with respect to the Killing form on $\mathfrak{g}$. It is therefore sufficient to show that h_k is in the kernel of $(ad\,e_0^\vee)^{m_k+1}$ in the algebra $\mathfrak{g}^\vee$, where $(e_0^\vee, 2\rho, f_0^\vee)$ is the canonical principal $\mathfrak{sl}_2$-triple of $\mathfrak{g}^\vee$. Here m_k is the kth exponent of $\mathfrak{g}^\vee$, which is the same as the kth exponent of $\mathfrak{g}$.

Putting $n = m_k + 1$ and $b = b_k$, it is enough to prove that for $b \in S^n(\mathfrak{g})^{\mathfrak{g}}$, the element $h = \iota_S(\rho)^{n-1}b$ of $\mathfrak{h}$ is in the kernel of $(ad\,e_0^\vee)^n$ in the algebra $\mathfrak{g}^\vee$. Let $\bar{b} = \Psi_0(b)$ denote the image of b under the Chevalley projection map $\Psi_0\colon S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{h})^W$. Clearly, $h = \iota_S(\rho)^{n-1}\bar{b}$. Now regard $\mathfrak{h}$ as the Cartan subalgebra of $\mathfrak{g}^\vee$, and let $\Psi_0^\vee\colon S(\mathfrak{g}^\vee)^{\mathfrak{g}^\vee} \to S(\mathfrak{h})^W$ be the Chevalley projection for $\mathfrak{g}^\vee$. Put $b^\vee = (\Psi_0^\vee)^{-1}(\bar{b})$ so that $b^\vee$ is the image of b under the algebra isomorphism

$$(\Psi_0^\vee)^{-1}\Psi_0\colon S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{g}^\vee)^{\mathfrak{g}^\vee},$$

and write $h = \iota_S(\rho)^{n-1} b^\vee$. Using the formula $(ad\, x)\iota_S(y) = \iota_S([x,y]) + \iota_S(y)\, ad\, x$ where $x, y \in \mathfrak{g}^\vee$ and both sides are operators on $S(\mathfrak{g}^\vee)$, we can now write

$$(ad\, e_0^\vee)^n \iota_S(\rho)^{n-1} b^\vee = \sum \frac{n!}{d_1! \dots d_n!} \iota_S((ad\, e_0^\vee)^{d_1}\rho) \dots \iota_S((ad\, e_0^\vee)^{d_{n-1}}\rho)(ad\, e_0^\vee)^{d_n} b^\vee.$$

The sum on the right is over all $n$tuples $d_1, \dots, d_n$ of non-negative integers, such that $d_1 + \dots + d_n = n$. Observe that for each such $n$tuple, either $d_i \ge 2$ for some $i < n$ so that $(ad\, e_0^\vee)^{d_i}\rho = 0$ (remembering the key $\mathfrak{sl}_2$ relation $(ad\, e_0^\vee)^2\rho = 0$); or else $d_n \ge 1$, so that $(ad\, e_0^\vee)^{d_n} b^\vee = 0$ because $b^\vee$ is $ad\, \mathfrak{g}^\vee$-invariant. It follows that the right-hand side of the last equation is zero, as required. □

References

[1] Alekseev, A., Meinrenken, E., Woodward, C. Group-valued equivariant localization, Invent. Math. 140 (2000), 327–350.

[2] Andruskiewitsch, N.; Schneider, H.-J. Finite Quantum Groups and Cartan Matrices, Adv. Math. 154 (2000), 1–45.

[3] Bazlov, Y. Exterior powers of the adjoint representation of a semisimple Lie algebra, PhD thesis, The Weizmann Institute of Science (2003).

[4] Bazlov, Y. Nichols-Woronowicz algebra model for Schubert calculus on Coxeter groups, J. Algebra 297 (2006), no. 2, 372–399.

[5] Bazlov, Y., Berenstein, A. Braided doubles and rational Cherednik algebras, Adv. Math. 220 (2009), no. 5, 1466–1530.

[6] Bourbaki, N. Groupes et algèbres de Lie, Ch. IV, V, VI. Hermann, 1968.

[7] Chevalley, C. The Betti numbers of the exceptional Lie groups, in: Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Providence, American Mathematical Society, 1952, vol. 2, pp. 21–24.

[8] Chevalley, C. The Algebraic Theory of Spinors. Coulmbia Univ. Press, New York, 1954.

[9] Chevalley, C. Invariants of finite groups generated by reflections, Amer. J. Math. 67 (1955), 778–782.

[10] Dixmier, J. Enveloping algebras. AMS, Providence, RI, 1996.

[11] Duflo, M. Caractères des algèbres de Lie résolubles, C. R. Acad. Sci. Paris Sér. A–B 269 (1969), A437–A438.

[12] Elashvili, A. G., Kac, V. G. Classification of good gradings of simple Lie algebras, in: Lie groups and invariant theory, 85–104, Amer. Math. Soc. Transl. Ser. 2, 213, Amer. Math. Soc., Providence, RI, 2005.

[13] Geck, M., Pfeiffer, G. Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras. Oxford University Press, 2000.

[14] Greenstein, J., Joseph, A. A Chevalley-Kostant presentation of basic modules for $\widehat{\mathfrak{sl}_2}$ and the associated affine KPRV determinants at $q = 1$, Bull. Sci. Math. 125 (2001), no. 2, 1–23.

[15] Harish-Chandra, On some applications of the universal enveloping algebra of a semisimple Lie algebra, Trans. Amer. Math. Soc. 70 (1951), 28–96.

[16] Joseph, A. On an affine quantum KPRV determinant at $q = 1$, Bull. Sci. Math. 125 (2001), no. 1, 23–48.

[17] Kostant, B. The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie group, Amer. J. Math. 81 (1959), no. 4, 973–1032.

[18] Kostant, B. Lie Group Representations on Polynomial Rings, Amer. J. Math. 85 (1963), no. 3, 327–404.

[19] Kostant, B. Clifford Algebra Analogue of the Hopf-Koszul-Samelson Theorem, the ρ-Decomposition $C(\mathfrak{g}) = \operatorname{End} V_\rho \otimes C(P)$, and the $\mathfrak{g}$-Module Structure of $\bigwedge\mathfrak{g}$. Adv. Math. 125 (2), 1997, 275–350.

[20] Kostant, B., private communication.

[21] Koszul, J. L. Homologie et cohomologie des algèbres de Lie, Bull. Soc. Math. Fr. 78, 1950, 65–127.

[22] Nichols, W. D. Bialgebras of type one, Comm. Algebra 6 (1978), no. 15, 1521–1552.

[23] Rohr, R. P. Principal basis in Cartan subalgebra, preprint arXiv:0804.3289.

[24] Shephard, G. C., Todd, J. A., Finite unitary reflection groups. Canadian J. Math. 6 (1954), 274–304.

[25] Tanisaki, T. Harish-Chandra isomorphisms for quantum algebras. Comm. Math. Phys. 127 (1990), no. 3, 555–571.

[26] Wallach, N. R.; Hunziker, M. On the Harish-Chandra homomorphism of invariant differential operators on a reductive Lie algebra. Representation theory and harmonic analysis (Cincinnati, OH, 1994), 223–243, Contemp. Math., 191, Amer. Math. Soc., Providence, RI, 1995.

[27] Woronowicz, S. L. Differential calculus on compact matrix pseudogroups (quantum groups). Comm. Math. Phys. 122 (1989), no. 1, 125–170.

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom
E-mail address: `y.bazlov@warwick.ac.uk`